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Shape Criteria of Bernstein-Bézier Polynomials over Simplexes

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Abstract—This paper discusses the criteria of convexity, monotonicity, and positivity of Bernstein-Bézier polynomials over simplexes.

Keywords—Bernstein-Bézier polynomials, Bernstein-Bézier nets, Convexity, Monotonicity, Positivity.

1. INTRODUCTION

Since the Bernstein-Bézier representation of a piecewise function is based on a local coordinate system, the barycentric coordinates, it is invariant under linear transformations and gives simple expressions of the smoothing conditions. Hence, it is useful in computer-aided design, finite element analysis, approximation theory, and wavelet analysis. In this paper, we discuss some shape criteria of Bernstein-Bézier polynomials of several variables. Section 2 gives the convexity criteria of Bernstein-Bézier polynomials defined on an arbitrary simplex. In Section 3, we discuss the criteria of monotonicity and positivity of Bernstein-Bézier polynomials over simplexes. First, we give some notions and some properties about the basis functions of Bernstein-Bézier polynomials.

Let $\mathbf{x}^0, \dots, \mathbf{x}^s \in \mathbb{R}^s$, $s \geq 1$, $\mathbf{x}^i = (x_1^i, \dots, x_s^i)$ and consider the convex hull

$$\langle \mathbf{x}^0, \dots, \mathbf{x}^s \rangle = \left\{ \sum_{i=0}^{s} \alpha_i \mathbf{x}^i : \sum_{i=0}^{s} \alpha_i = 1, \alpha_i \geq 0 \right\}.$$

This convex hull is called an s-simplex if its signed volume

$$\mathrm{Vol}_s \langle \mathbf{x}^0, \dots, \mathbf{x}^s \rangle = \frac{1}{s!} \begin{vmatrix} 1 & x_1^0 & \cdots & x_s^0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1^s & \cdots & x_s^s \end{vmatrix}$$

is nonzero. Suppose that $\langle \mathbf{x}^0, \dots, \mathbf{x}^s \rangle$ is an s-simplex. Then, any $\mathbf{x} \in \mathbb{R}^s$ can be identified by an $(s+1)$-tuple $\lambda = (\lambda_0, \dots, \lambda_s)$ where

$$\lambda_i = \lambda_i(\mathbf{x}) = \frac{\mathrm{Vol}_s \langle \mathbf{x}^0, \dots, \mathbf{x}^{i-1}, \mathbf{x}, \mathbf{x}^{i+1}, \dots, \mathbf{x}^s \rangle}{\mathrm{Vol}_s \langle \mathbf{x}^0, \dots, \mathbf{x}^s \rangle}.$$

This $(s+1)$-tuple is called the barycentric coordinates of $\mathbf{x}$ relative to the s-simplex $\langle \mathbf{x}^0, \dots, \mathbf{x}^s \rangle$. Thus, each $\lambda_i = \lambda_i(\mathbf{x})$ is a linear polynomial in $\mathbf{x}$ with $\sum_{i=0}^{s} \lambda_i = 1$, and if $\mathbf{x} \in \langle \mathbf{x}^0, \dots, \mathbf{x}^s \rangle$, then $\lambda_i \geq 0$.

For any $\beta = (\beta_0, \dots, \beta_s) \in \mathbf{Z}_+^{s+1}$, and $n \in \mathbf{Z}_+$, we will use the usual multivariate notation

$$\lambda^\beta = \lambda_0^{\beta_0} \cdots \lambda_s^{\beta_s}, \qquad \beta! = \beta_0! \cdots \beta_s!, \qquad |\beta| = \beta_0 + \cdots + \beta_s.$$

Hence,

$$\phi^n_\beta(\lambda) := \frac{n!}{\beta!}\lambda^\beta, \qquad |\beta| = \beta_0 + \cdots + \beta_s, \tag{1}$$

is a polynomial in $\pi^s_{|\beta|}$, the space of all polynomials in one variable of order $|\beta|+1$, or degree at most $|\beta|$. In fact, it is easy to see that for any positive integer n, $\{\phi^n_\beta(\lambda) : |\beta| = n\}$ is a basis of the polynomial space π^s_n. With any set $\{\alpha^n_\beta\} = \{\alpha^n_\beta\}_{\beta\in \mathbf{Z}^{s+1}_+, |\beta|=n} \subset \mathbb{R}$, one may associate the polynomial

$$p_n(\mathbf{x}) = B_n[\{\alpha^n_\beta\};\lambda] = \sum_{|\beta|=n} \alpha^n_\beta \phi^n_\beta(\lambda), \tag{2}$$

which is called a Bernstein-Bézier polynomial of total degree n relative to the s-simplex $\langle \mathbf{x}^0,\dots,\mathbf{x}^s\rangle$. In addition, $\{\alpha^n_\beta : |\beta| = n\}$ in (2) is called the set of Bézier coefficients of the polynomial p_n. And the set

$$\left\{\left(\frac{\beta_0}{n}\mathbf{x}^0 + \cdots + \frac{\beta_s}{n}\mathbf{x}^s,\ \alpha^n_\beta\right) : |\beta| = n\right\}$$

is called the Bézier net of the polynomial p_n.

For the basis functions $\phi^n_\beta(\lambda)$ of Bernstein-Bézier polynomials defined by (1), we have the following result.

LEMMA 1. *For functions $\phi^n_\beta(\lambda)$ defined by (1), the following two inequalities hold:*

$$0 \le \phi^n_\beta(\lambda) \le \frac{(n-1)!}{\beta! n^{n-1}}\left(\sum_{i=0}^s \beta_i\lambda_i\right)^n, \tag{3}$$

$$0 \le \phi^n_\beta(\lambda) \le \frac{(n-1)!}{\beta!}\sum_{i=0}^s \beta_i\lambda_i^n. \tag{4}$$

PROOF. Inequalities (3) and (4) can be deduced from the mean inequalities

$$\prod_{k=1}^m a_k^{u_k} \le \left(\frac{\sum_{k=1}^m u_k a_k}{\sum_{k=1}^m u_k}\right)^{\sum_{k=1}^m u_k}$$

and

$$\prod_{k=1}^m a_k^{v_k} \le \sum_{k=1}^m a_k v_k$$

directly, where m is any positive integer, $a_k \ge 0$, $u_k, v_k > 0$, and $\sum_{k=1}^m v_k = 1$ (cf., [1]). Obviously, the equal signs in (3) and (4) hold if and only if all the $\{\lambda_k\}_{k=0}^s$ are equal.

In the following, we will use the notation

$$D_{\mathbf{y}} = \sum_{i=1}^s y_i \frac{\partial}{\partial x_i}, \tag{5}$$

where $\mathbf{x} = (x_1,\dots,x_s)$ and $\mathbf{y} = (y_1,\dots,y_s)$. For $\mathbf{y} = \mathbf{x}^i - \mathbf{x}^j$, we write

$$D_{ij} = D_{\mathbf{y}} = D_{\mathbf{x}^i - \mathbf{x}^j}, \qquad i \ne j. \tag{6}$$

By using the barycentric coordinates $\{\lambda_\ell\}_{\ell=0}^s$ of $\mathbf{x}\in\mathbb{R}^s$ relative to an s-simplex $T_s = \langle \mathbf{x}^0,\dots,\mathbf{x}^s\rangle$, we also write

$$\mathbf{x} = \sum_{\ell=0}^s \lambda_\ell \mathbf{x}^\ell.$$

We also need the notation

$$E_i a_\alpha = a_{\alpha+\mathbf{e}^i}, \tag{7}$$

and

$$\Delta_{ij} a_\alpha^n = E_i a_\alpha^n - E_j a_\alpha^n, \tag{8}$$

where $\mathbf{e}^i = (\delta_{ij})_{j=0}^s$ denotes the i^{th} coordinate vector in $\mathbb{R}^{s+1}$.

Thus, we have

$$\begin{aligned}(D_{ij} p_n)(\mathbf{x}) &= n \sum_{|\alpha|=n-1} (E_i - E_j) a_\alpha^n \phi_\alpha^{n-1}(\lambda)\\ &= n \sum_{|\alpha|=n-1} \Delta_{ij} a_\alpha^n \phi_\alpha^{n-1}(\lambda).\end{aligned} \tag{9}$$

Now we discuss the smoothness conditions for two adjacent simplices. A simplex S_k with k+1 vertices $\mathbf{x}^0, \dots, \mathbf{x}^k$ in $\mathbb{R}^s$ is denoted by $S_k = \langle \mathbf{x}^0, \dots, \mathbf{x}^k \rangle$, $0 \le k < s$, and S_k is also called a k-simplex if $\mathrm{Vol}_k S_k > 0$ for $k > 0$. S_0 is a point, and we will also call S_0 a zero-simplex for convenience. If $S = \langle \mathbf{x}^0, \dots, \mathbf{x}^s \rangle$ is an s-simplex, then for each k, $0 \le k < s$, a k-simplex $\langle \mathbf{x}^{i_0}, \dots, \mathbf{x}^{i_k} \rangle$, $0 \le i_0 < \dots < i_k \le s$, is also called a k-facet of S.

For any direction V, there exists a vector $\hat{\mathbf{c}}_V = (c_0, c_1, \dots, c_s)$ in

$$\hat{C}_0 = \{\hat{c} \in \mathbb{R}^{s+1} : c_0 + \dots + c_s = 0\},$$

relative to s-simplex $T_s = \langle \mathbf{x}^0, \dots, \mathbf{x}^s \rangle$, such that

$$V = \sum_{i=0}^{s} c_i \mathbf{x}^i. \tag{10}$$

Thus, from (9), we have

$$\begin{aligned}D_V p_n &= n \sum_{|\alpha|=n-1} \left(\sum_{i=0}^{s} c_i E_i a_\alpha^n \right) \phi_\alpha^{n-1}(\lambda)\\ &= n \sum_{|\alpha|=n-1} \hat{\mathbf{c}}_V^\top \cdot \hat{\mathbf{b}}_\alpha \phi_\alpha^{n-1}(\lambda),\end{aligned} \tag{11}$$

where $\hat{\mathbf{b}}_\alpha = (E_i a_\alpha^n)_{i=0}^s$, $|\alpha| = n-1$.

Similarly, we have

$$\begin{aligned}D_V^2 p_n &= D_V(D_V p_n)\\ &= n(n-1) \sum_{|\alpha|=n-2} \hat{\mathbf{c}}_V^\top \hat{Q}_{\alpha,T_s} \hat{\mathbf{c}}_V \phi_\alpha^{n-2}(\lambda),\end{aligned} \tag{12}$$

where

$$\hat{Q}_{\alpha,T_s} := (E_i E_j a_\alpha^n)_{i,j=0}^{s,s}, \qquad |\alpha| = n-2$$

is an $(s+1) \times (s+1)$ square matrix.

Since (10) can be written as

$$V = \sum_{i=1}^{s} c_i \left(\mathbf{x}^i - \mathbf{x}^0 \right), \tag{13}$$

equations (11) and (12) can be reformulated as

$$D_V p_n = n \sum_{|\alpha|=n-1} \mathbf{c}_V^\top \mathbf{b}_\alpha \phi_\alpha^{n-1}(\lambda), \tag{14}$$

and

$$D_V^2 p_n = n(n-1) \sum_{|\alpha|=n-2} \mathbf{c}_V^\top Q_{\alpha,T_s} \mathbf{c}_V \phi_\alpha^{n-2}(\lambda), \tag{15}$$

respectively, where $c_V = (c_1, \ldots, c_s)^\top$, $\mathbf{b}_\alpha = (\Delta_{i0} a_\alpha^n)_{i=1}^s$, and

$$Q_{\alpha,T_s} := (\Delta_{i0}\Delta_{j0} a_\alpha^n)_{i,j=1}^{s,s}, \tag{16}$$

for $|\alpha| = n-2$.

2. CONVEXITY CRITERIA FOR BERNSTEIN-BÉZIER POLYNOMIALS OVER A SIMPLEX

It is well known, that for a piecewise C^1 (i.e., continuously differentiable) function to be convex, it is necessary and sufficient that all its pieces are convex. Thus, in order to consider the convexity of the Bernstein-Bézier representation of a C^1 spline function or a C^1 finite element, we only need to derive convexity criteria for a Bernstein-Bézier polynomial over a simplex.

Let $p_n \in \pi_n^s$. Obviously, p_n is convex on T if and only if $D_\mathbf{V}^2 p_n(\mathbf{x}) \geq 0$ for any direction vector $\mathbf{V}$ and any point $\mathbf{x} \in T$. From (15), for any fixed direction $\mathbf{V}$, if $\mathbf{c}_\mathbf{V}^\top Q_{\mathbf{k},T_s} \mathbf{c}_\mathbf{V} \geq 0$, then $D_\mathbf{V}^2 p_n(\mathbf{x}) \geq 0$. On the other hand, for quadratic polynomials p_2, if $D_\mathbf{V}^2 p_2(\mathbf{x}) \geq 0$, then $\mathbf{c}_\mathbf{V}^\top Q_{T_s} \mathbf{c}_\mathbf{V} = (1/2) D_\mathbf{V}^2 p_2(\mathbf{x}) \geq 0$, where $Q_{T_s} := Q_{\mathbf{0},T_s}$; and for cubic polynomials p_3 if $D_\mathbf{V}^2 p_n(\mathbf{x}) \geq 0$, then $\mathbf{c}_\mathbf{V}^\top Q_{\mathbf{k},T_s} \mathbf{c}_\mathbf{V} \geq 0$, where $\mathbf{k} = (1,0,0), (0,1,0)$, and $(0,0,1)$. Thus, we have the following lemma.

LEMMA 2. *If $Q_{\mathbf{k},T_s}$, $|\mathbf{k}| = n-2$, defined in* (15) *and* (16) *is positive semidefinite, then $p_n(\mathbf{x})$ is convex. In addition, $p_2(\mathbf{x})$ and $p_3(\mathbf{x})$ are convex if and only if Q_{T_s} and $Q_{\mathbf{k},T_s}$, $|\mathbf{k}| = 1$, are positive semidefinite, respectively.*

Recall that an $(s+1) \times (s+1)$ symmetric matrix $A = [a_{ij}]_{1 \leq i,j \leq s+1}$ is called conditionally positive definite if

$$(\hat{\mathbf{c}},\ A\hat{\mathbf{c}}) = \sum_{1 \leq i,j \leq s+1} c_i c_j a_{ij} \geq 0,$$

for all $\hat{\mathbf{c}} = (c_1, \ldots, c_{s+1})^\top \in \hat{C}_0 = \{\hat{\mathbf{c}} \in \mathbb{R}^{s+1} : c_1 + \cdots + c_{s+1} = 0\}$. Hence, from (12), we also have the following lemma.

LEMMA 3. *If $\hat{Q}_{\mathbf{k},T_s}$, $|\mathbf{k}| = n-2$, defined in* (12) *is conditionally positive definite, then $p_n(\mathbf{x})$ is convex. In addition, $p_2(\mathbf{x})$ and $p_3(\mathbf{x})$ are convex if and only if $\hat{Q}_{\mathbf{0}T_s}$ and $\hat{Q}_{\mathbf{k},T_s}$, $|\mathbf{k}| = 1$, are conditionally positive definite, respectively.*

The conditions for convexity given by Lemmas 2 and 3 are complicated in a higher dimensional setting. Thus, we need some simpler, but perhaps a little stronger convexity criteria. We start our analysis with dimension two.

Chang and Davis [2] once gave $Q_{\mathbf{k}T}$, $T = T_2$ by using Taylor's formula. They also gave a sufficient condition which guarantees that $Q_{\mathbf{k},T}$ is positive semidefinite, namely

$$\begin{aligned} \Delta_\mathbf{k}^{(1)} &:= \Delta_{10}\Delta_{20} a_\mathbf{k} \geq 0, \\ \Delta_\mathbf{k}^{(2)} &:= \Delta_{21}\Delta_{01} a_\mathbf{k} \geq 0, \\ \Delta_\mathbf{k}^{(3)} &:= \Delta_{02}\Delta_{12} a_\mathbf{k} \geq 0. \end{aligned} \tag{17}$$

They also prove that (17) is a necessary and sufficient convexity condition for the Bézier net of p_n. (Note that convexity of a polynomial does not necessarily imply the convexity of its Bézier net.)

By using the notation in (17), we have

$$\begin{aligned} \Delta_{10}^2 a_\mathbf{k} &= \Delta_\mathbf{k}^{(1)} + \Delta_\mathbf{k}^{(2)}, \\ \Delta_{20}^2 a_\mathbf{k} &= \Delta_\mathbf{k}^{(1)} + \Delta_\mathbf{k}^{(3)}, \end{aligned}$$

and

$$Q_{\mathbf{k},T} = \begin{bmatrix} \Delta_{\mathbf{k}}^{(1)} + \Delta_{\mathbf{k}}^{(2)} & \Delta_{\mathbf{k}}^{(1)} \\ \Delta_{\mathbf{k}}^{(1)} & \Delta_{\mathbf{k}}^{(1)} + \Delta_{\mathbf{k}}^{(3)} \end{bmatrix}. \tag{18}$$

Condition (17) was weakened by Chang and Feng [3] to be

$$\begin{aligned} &\Delta_{\mathbf{k}}^{(1)} + \Delta_{\mathbf{k}}^{(2)} \ge 0, \quad \Delta_{\mathbf{k}}^{(2)} + \Delta_{\mathbf{k}}^{(3)} \ge 0, \quad \Delta_{\mathbf{k}}^{(3)} + \Delta_{\mathbf{k}}^{(1)} \ge 0, \\ &\Delta_{\mathbf{k}}^{(1)}\Delta_{\mathbf{k}}^{(2)} + \Delta_{\mathbf{k}}^{(2)}\Delta_{\mathbf{k}}^{(3)} + \Delta_{\mathbf{k}}^{(3)}\Delta_{\mathbf{k}}^{(1)} \ge 0. \end{aligned} \tag{19}$$

From Lemma 2, condition (19) is equivalent to the condition that the matrix Q_T is positive semidefinite. It is clear that (17) implies condition (19) is sharp for $n = 2, 3$, since it is a necessary and sufficient condition for positive semidefiniteness of $Q_{\mathbf{k},T}$, $|\mathbf{k}| = 0, 1$. That is, by Lemma 2, it is a necessary and sufficient condition for convexity of p_2 and p_3. Unfortunately, condition (19) is nonlinear. In the following, we will give a new criterion which is linear, but is superior to condition (17).

From (19), one expects at most one of $\Delta_{\mathbf{k}}^{(i)}$, $i = 1, 2, 3$, could be negative. Hence, the following theorem gives a criterion on convexity when either all $\Delta_{\mathbf{k}}^{(i)}$, $i = 1, 2, 3$, are nonnegative, or one of them is negative.

THEOREM 4. *If either* (17) *or the following condition* (20) *is satisfied, then the Bernstein-Bézier polynomial* $p_n = B_n[\{a_{\mathbf{k}}\}; \lambda]$ *is convex. Here,*

$$\begin{aligned} &\Delta_{\mathbf{k}}^{(u)} \le 0, \\ &2\Delta_{\mathbf{k}}^{(u)} + \Delta_{\mathbf{k}}^{(v)} \ge 0, \\ &2\Delta_{\mathbf{k}}^{(u)} + \Delta_{\mathbf{k}}^{(w)} \ge 0, \end{aligned} \tag{20}$$

where (u, v, w) *is a permutation of* (1,2,3).

PROOF. Obviously, the condition in Theorem 4 implies condition (19). In fact, if either condition (17) or condition (20) holds, then

$$\Delta_{\mathbf{k}}^{(u)} + \Delta_{\mathbf{k}}^{(v)} \ge |\Delta_{\mathbf{k}}^{(u)}| \ge 0,$$

or

$$\Delta_{\mathbf{k}}^{(u)} + \Delta_{\mathbf{k}}^{(w)} \ge |\Delta_{\mathbf{k}}^{(u)}| \ge 0.$$

Thus, we have

$$\Delta_{\mathbf{k}}^{(v)} + \Delta_{\mathbf{k}}^{(w)} \ge 2\left[|\Delta_{\mathbf{k}}^{(u)}| - \Delta_{\mathbf{k}}^{(u)}\right] \ge 0,$$

and

$$\left(\Delta_{\mathbf{k}}^{(u)} + \Delta_{\mathbf{k}}^{(v)}\right)\left(\Delta_{\mathbf{k}}^{(u)} + \Delta_{\mathbf{k}}^{(w)}\right) \ge \left[\Delta_{\mathbf{k}}^{(u)}\right]^2.$$

Hence, condition (19) is satisfied, and p_n is convex.

It is obvious that the condition shown in Theorem 4 is linear and weaker than Chang and Davis' condition. In addition, this condition can also be preserved by the so-called parallel subdivision.

Let $\mathbf{x}_*^1$, $\mathbf{x}_*^2$, and $\mathbf{x}_*^3$ be the vertices of subtriangle $T^* = T_2^*$ that lies on the plane determined by the original triangle $T = T_2$. Following Gregory and Zhou [4], we have the following definition.

Definition 5. *A triangle T^* is said to be parallel to T, if there exists a nonzero scalar ρ and a permutation (i_1, i_2, i_3) of $\{1, 2, 3\}$, such that*

$$\mathbf{x}_*^{i_u} - \mathbf{x}_*^{i_v} = \rho(\mathbf{x}^u - \mathbf{x}^v), \tag{21}$$

where $u, v = 1, 2, 3$, $u \neq v$.

Assume that $\mathbf{x}_*^i$ has barycentric coordinates $(\lambda_1^i, \lambda_2^i, \lambda_3^i)$ with respect to T; that is,

$$\mathbf{x}_*^i = \lambda_1^i \mathbf{x}^1 + \lambda_2^i \mathbf{x}^2 + \lambda_3^i \mathbf{x}^3, \qquad \lambda_1^i + \lambda_2^i + \lambda_3^i = 1. \tag{22}$$

It is obvious that if T^* is parallel to T, then none of the sets $\{\lambda_1^1, \lambda_1^2, \lambda_1^3\}$, $\{\lambda_2^1, \lambda_2^2, \lambda_2^3\}$, and $\{\lambda_3^1, \lambda_3^2, \lambda_3^3\}$ has all distinct numbers.

Now, we make the following definition of convexity-preserving conditions.

Definition 6. *A subtriangle T^* of T is said to preserve a given convexity condition on T if this condition is satisfied by every restricted Bernstein-Bézier polynomials p_n with respect to T^* whenever it is satisfied by p_n on T.*

From Goodman [5], if T^* is a subtriangle of T obtained from a mid-point subdivision process, then T^* preserves the Chang and Davis' condition (17). From Gregory and Zhou [4], if T^* is a subtriangle of T, then T^* preserves Chang and Davis' condition (17) if and only if it is parallel to T. Now we are going to prove the following result.

Theorem 7. *Let T^* be a subtriangle of T. Then, T^* preserves the convexity condition shown in Theorem 4 if and only if it is parallel to T.*

Proof. (Sufficiency) $B_n[\{a_i\}; \lambda]$ satisfies the convexity condition (17) or (20) on T, and T^* is a parallel subtriangle of T. Let $T_s^* = \langle \mathbf{x}_*^0, \dots, \mathbf{x}_*^s \rangle$ be a subsimplex of $T_s = \langle \mathbf{x}^0, \dots, \mathbf{x}^s \rangle$, λ^* and λ be barycentric coordinates of a point in T_s^* with respect to T_s^* and T_s, respectively. If $B_n^*[\{b_\beta\}; \lambda^*]$ is the Bernstein-Bézier representation of $B_n[\{a_\alpha\}; \lambda]$ with respect to T^*, then, $B_n[\{a_\alpha\}; \lambda]$ and $B_n^*[\{b_\beta\}; \lambda^*]$ can be written as

$$B_n[\{a_\alpha\}, \lambda] = \left(\sum_{j=0}^{s} \lambda_j E_j \right)^n a_{0,\dots,0},$$

and

$$B_n^*[\{b_\beta\}, \lambda^*] = \left(\sum_{j=0}^{s} \lambda_j^* E_j^* \right)^n b_{0,\dots,0}$$

by using the notation $E_i a_\alpha = a_{\alpha + \mathbf{e}^i}$ and $E_i^* b_\beta = b_{\beta + \mathbf{e}^i}$, respectively. Assume that $\mathbf{x}_*^i$ has barycentric coordinates $(\lambda_0^i, \dots, \lambda_s^i)$ with respect to T_s, then

$$\lambda_i = \sum_{j=0}^{s} \lambda_j^* \lambda_i^j,$$

for $i = 0, 1, \dots, s$. Substituting this relation to $B_n[\{a_\alpha\}; \lambda]$, it is very easy to obtain

$$E_i^* = \sum_{j=0}^{s} \lambda_j^i E_j, \qquad i = 0, 1, \dots, s,$$

$$b_\beta = \prod_{i=0}^{s} \left(\sum_{j=0}^{s} \lambda_j^i E_j \right)^{\beta_i} a_{0,\dots,0},$$

for all $\beta \in \mathbf{Z}_0^{s+1}$ with $|\beta| = n$. The last equation is also called the subdivision recurrence relation. If T_s^* is a mid-point subtriangle, the corresponding subdivision recurrence relation was given by [6]. By using these relations and Definition 5,

$$\begin{aligned}\Delta_{\mathbf{i}}^{(u)^*} &:= (E_u^* - E_v^*)(E_u^* - E_w^*)b_{\mathbf{i}} \\ &= \prod_{u=1}^{3}\left(\sum_{v=1}^{3} \lambda_v^u E_v\right)^{i_u} \rho^2 \Delta_{\mathbf{0}}^{(u)} \\ &= \sum_{|\mathbf{r}|=i_1} \sum_{|\mathbf{s}|=i_2} \sum_{|\mathbf{t}|=i_3} \left(\rho^2 \phi_{\mathbf{r}}^{i_1}(\lambda^1)\, \phi_{\mathbf{s}}^{i_2}(\lambda^2)\, \phi_{\mathbf{t}}^{i_3}(\lambda^3)\right) \Delta_{\mathbf{i}}^{(u)},\end{aligned}$$

and

$$\begin{aligned}2\Delta_{\mathbf{i}}^{(u)^*} + \Delta_{\mathbf{i}}^{(v)^*} &:= \left[2(E_u^* - E_v^*)(E_u^* - E_w^*) + (E_v^* - E_u^*)(E_v^* - E_w^*)\right] b_{\mathbf{i}} \\ &= \sum_{|\mathbf{r}|=i_1} \sum_{|\mathbf{s}|=i_2} \sum_{|\mathbf{t}|=i_3} \left(\rho^2 \phi_{\mathbf{r}}^{i_1}(\lambda^1)\, \phi_{\mathbf{s}}^{i_2}(\lambda^2)\, \phi_{\mathbf{t}}^{i_3}(\lambda^3)\right) \left(2\Delta_{\mathbf{i}}^{(u)} + \Delta_{\mathbf{i}}^{(v)}\right),\end{aligned}$$

where $u = 1, 2,$ or 3, (u, v, w) is a permutation of (1,2,3). Since $\phi_{\mathbf{r}}^{i_1}(\lambda^1)$, $\phi_{\mathbf{s}}^{i_2}(\lambda^2)$, and $\phi_{\mathbf{t}}^{i_3}(\lambda^3)$ are nonnegative, we conclude that $B_n^*[\{b_{\mathbf{i}}\}; \lambda^*]$, the restriction of $B_n[\{a_{\mathbf{i}}\}; \lambda]$ on T^*, also satisfies the same convexity condition (17) or (20).

(Necessity) We assume that T^* is not parallel to T. From Definition 5, without loss of generality, we may set $\lambda_1^2 > \lambda_1^1 > \lambda_1^3 \geq 0$. Consider $B_n[\{a_{\mathbf{i}}\}; \lambda]$ with Bézier coefficients

$$a_{\mathbf{i}} = \delta_{i_1,n}, \qquad |\mathbf{i}| = n. \tag{23}$$

Obviously, $B_n[\{a_{\mathbf{i}}; \lambda]$ satisfies the convexity condition (17). By using the subdivision recurrence relation,

$$\begin{aligned}\Delta_{(0,n-2,0)}^{(1)^*} &= \left(\lambda_1^2\right)^{n-2}\left(\lambda_1^1 - \lambda_1^2\right)\left(\lambda_1^1 - \lambda_1^3\right) < 0, \\ 2\Delta_{(0,n-2,0)}^{(1)^*} + \Delta_{(0,n-2,0)}^{(2)^*} &= \left(\lambda_1^2\right)^{n-2}\left(\lambda_1^2 - \lambda_1^1\right)\left(\lambda_1^2 + \lambda_1^3 - 2\lambda_1^1\right), \\ 2\Delta_{(0,n-2,0)}^{(1)^*} + \Delta_{(0,n-2,0)}^{(3)^*} &= \left(\lambda_1^2\right)^{n-2}\left(\lambda_1^1 - \lambda_1^3\right)\left(2\lambda_1^1 - \lambda_1^2 - \lambda_1^3\right).\end{aligned}$$

If $B_n^*[\{b_{\mathbf{i}}\}; \lambda^*]$ satisfies the convexity condition (20), then there must be

$$\lambda_1^1 = \frac{\lambda_1^2 + \lambda_1^3}{2}. \tag{24}$$

On the other hand, we consider

$$a_{\mathbf{i}} = \delta_{i_2,n}, \qquad \mathbf{i} \in \mathbf{Z}_0^3, \qquad |\mathbf{i}| = n.$$

Thus, we have

$$\begin{aligned}\Delta_{\mathbf{i}}^{(1)^*} &= \left(\lambda_2^1\right)^{i_1}\left(\lambda_2^2\right)^{i_2}\left(\lambda_2^3\right)^{i_3}\left(\lambda_2^1 - \lambda_2^2\right)\left(\lambda_2^1 - \lambda_2^3\right), \\ \Delta_{\mathbf{i}}^{(2)^*} &= \left(\lambda_2^1\right)^{i_1}\left(\lambda_2^2\right)^{i_2}\left(\lambda_2^3\right)^{i_3}\left(\lambda_2^2 - \lambda_2^3\right)\left(\lambda_2^2 - \lambda_2^1\right), \\ \Delta_{\mathbf{i}}^{(3)^*} &= \left(\lambda_2^1\right)^{i_1}\left(\lambda_2^2\right)^{i_2}\left(\lambda_2^3\right)^{i_3}\left(\lambda_2^3 - \lambda_2^1\right)\left(\lambda_2^3 - \lambda_2^2\right),\end{aligned}$$

for all $\mathbf{i} \in \mathbf{Z}_0^3$, $|\mathbf{i}| = n$. If all $\Delta_{\mathbf{i}}^{(j)^*}$, $j = 1, 2, 3$, are nonnegative, then there must be $\lambda_2^1 = \lambda_2^2$, $\lambda_2^2 = \lambda_2^3$, or $\lambda_2^3 = \lambda_2^1$. By symmetry, we only need to consider the case $\lambda_2^1 = \lambda_2^2$. If one of $\Delta_{\mathbf{i}}^{(j)^*}$,

$j = 1,2,3$, is negative, we must prove it cannot be $\Delta_{\mathbf{i}}^{(1)^*}$ in order to avoid a contradiction. In fact, it is easy to see that

$$2\Delta_{\mathbf{i}}^{(1)^*} + \Delta_{\mathbf{i}}^{(2)^*} = \left(\lambda_2^1\right)^{i_1}\left(\lambda_2^2\right)^{i_2}\left(\lambda_2^3\right)^{i_3}\left(\lambda_2^2 + \lambda_2^3 - 2\lambda_2^1\right),$$

and

$$2\Delta_{\mathbf{i}}^{(1)^*} + \Delta_{\mathbf{i}}^{(3)^*} = \left(\lambda_2^1\right)^{i_1}\left(\lambda_2^2\right)^{i_2}\left(\lambda_2^3\right)^{i_3}\left(2\lambda_2^1 + \lambda_2^2 - \lambda_2^3\right).$$

In order to ensure the corresponding $B_n^*[\{b_{\mathbf{i}}\};\lambda]$ satisfies the convexity condition (20), then we must have

$$\lambda_2^1 = \frac{\lambda_2^2 + \lambda_2^3}{2}.$$

Note that by (24), the points $\mathbf{x}_*^1$, $\mathbf{x}_*^2$, and $\mathbf{x}_*^3$ are colinear, which is impossible. Thus, we only need to consider the situation when $\Delta_{\mathbf{i}}^{(2)^*}$ or $\Delta_{\mathbf{i}}^{(3)^*}$ is negative. Without loss of generality, we assume $\Delta_{\mathbf{i}}^{(2)^*} < 0$. By using a similar argument, we have

$$\lambda_2^2 = \frac{\lambda_2^1 + \lambda_2^3}{2}. \tag{25}$$

Finally, we consider

$$a_{\mathbf{i}} = \delta_{i_3,n}, \qquad \mathbf{i} \in \mathbf{Z}_0^3, \qquad |\mathbf{i}| = n.$$

Similarly, if all corresponding $\Delta_{\mathbf{i}}^{(j)^*}$, $j = 1,2,3$ are nonnegative, then it implies that $\lambda_3^1 = \lambda_3^2$, $\lambda_3^2 = \lambda_3^3$, or $\lambda_3^3 = \lambda_3^1$. Since we have assumed that $\lambda_2^1 = \lambda_2^2$, and $\lambda_1^2 > \lambda_1^1$, then $\lambda_3^1 \neq \lambda_3^2$. Out of the cases of $\lambda_3^2 = \lambda_3^3$ or $\lambda_3^3 = \lambda_3^1$, we only need to consider one by symmetry. Without loss of generality, we assume that $\lambda_3^2 = \lambda_3^3$. If one of $\Delta_{\mathbf{i}}^{(j)^*}$, $j = 1,2,3$ is negative, then from (24) and (25), we must have $\Delta_{\mathbf{i}}^{(3)^*} < 0$. Thus, in order to ensure that the corresponding $B_n^*[\{b_{\mathbf{i}}\};\lambda^*]$ satisfies the convexity condition (20), we must have

$$\lambda_3^3 = \frac{\lambda_3^1 + \lambda_3^2}{2}. \tag{26}$$

Hence, under the assumption of $\lambda_1^2 > \lambda_1^1 > \lambda_1^3 \geq 0$, we can summarize all the possible relations of λ_v^u, $u,v = 1,2,3$, that must be considered, as follows:

$$\begin{array}{llll}
\text{(i)} & \lambda_1^1 = \dfrac{\lambda_1^2 + \lambda_1^3}{2}, & \lambda_2^1 = \lambda_2^2, & \lambda_3^2 = \lambda_3^3; \\[2ex]
\text{(ii)} & \lambda_1^1 = \dfrac{\lambda_1^2 + \lambda_1^3}{2}, & \lambda_2^1 = \lambda_2^2, & \lambda_3^3 = \dfrac{\lambda_3^1 + \lambda_3^2}{2}; \\[2ex]
\text{(iii)} & \lambda_1^1 = \dfrac{\lambda_1^2 + \lambda_1^3}{2}, & \lambda_2^2 = \dfrac{\lambda_2^1 + \lambda_2^3}{2}, & \lambda_3^2 = \lambda_3^3; \\[2ex]
\text{(iv)} & \lambda_1^1 = \dfrac{\lambda_1^2 + \lambda_1^3}{2}, & \lambda_2^2 = \dfrac{\lambda_2^1 + \lambda_2^3}{2}, & \lambda_3^3 = \dfrac{\lambda_3^1 + \lambda_3^2}{2}.
\end{array}$$

It is not difficult to prove that under each situation (i)–(iv), there must exist a subtriangle T^* which does not satisfy the convexity condition in Theorem 4. This contradiction shows if T^* preserves the convexity condition in Theorem 4, it must be parallel to T. This completes the proof of Theorem 7.

An interesting question is that, when p_n does not satisfy the convexity condition in Theorem 4 on the original triangle T, does there exist a sub-triangulation of T, such that the restriction of p_n on each subtriangle satisfies this convexity condition? Goodman and the author [7] observed that this type of sub-triangulation exists. For example, for $p = p_2$, a sub-triangulation consisting of only two subtriangles is needed. However, when we consider the same problem for Chang and Davis' condition, a sub-triangulation consisting of seven subtriangles will be needed.

We now consider the special case $n = 2$, and $p = p_2 = B_2[\{a_{\mathbf{k}}\}; \lambda]$. Denote

$$B_{200} = \Delta_{\mathbf{0}}^{(1)}, \qquad B_{020} = \Delta_{\mathbf{0}}^{(2)}, \qquad B_{002} = \Delta_{\mathbf{0}}^{(3)}, \tag{27}$$

and

$$A_{110} = B_{200} + B_{020}, \qquad A_{101} = B_{200} + B_{002}, \qquad A_{011} = B_{020} + B_{002}. \tag{28}$$

Thus, $Q_T := Q_{\mathbf{0},T}$ defined by (16) can be written as

$$Q_T = \begin{bmatrix} A_{110} & B_{200} \\ B_{200} & A_{101} \end{bmatrix}. \tag{29}$$

It is easy to obtain

$$\det\ Q_T = \det \frac{\begin{bmatrix} \frac{\partial^2 p}{\partial x^2} & \frac{\partial^2 p}{\partial x \partial y} \\ \frac{\partial^2 p}{\partial x \partial y} & \frac{\partial^2 p}{\partial y^2} \end{bmatrix}}{\delta_T^2},$$

where δ_T is twice the area of triangle T. If Q_T is positive semidefinite, then the Hessian matrix of p is positive semidefinite. This gives another proof of the result about $p = p_2$ in Lemma 2.

Let p be a Bernstein-Bézier polynomial defined on T. Then, from the condition (17), p is convex on T if

$$B_{200} \geq 0, \qquad B_{020} \geq 0, \qquad B_{002} \geq 0,$$

where B_{200}, B_{020}, and B_{002} are defined by (17) and (27). In fact, it is easy to see that

$$\det\ Q_T = B_{200}B_{020} + B_{020}B_{002} + B_{002}B_{200},$$

where δ_T is twice the area of the triangle T. Note that $A_{110} = B_{200} + B_{020} \geq 0$. Thus, Q_T is positive semidefinite. By Lemma 2, p is then convex.

From Theorem 4, we immediately have the following result.

COROLLARY 8. *Let p be a Bernstein-Bézier polynomial defined on T. Then, p is convex on T if*

$$A_{110} \geq |B_{200}| \qquad \text{and} \qquad A_{101} \geq |B_{200}|,$$

or equivalently, if either

$$B_{200} \geq 0, \qquad B_{020} \geq 0, \qquad \text{and} \qquad B_{002} \geq 0;$$

or

$$B_{200} \leq 0, \qquad 2B_{200} + B_{020} \geq 0, \qquad \text{and} \qquad 2B_{200} + B_{002} \geq 0$$

holds. Here, B_{200}, B_{020}, B_{002}, A_{110}, and A_{101} are defined in (17),(27), *and* (28).

Next, we consider the convexity criteria for Bernstein-Bézier polynomials over a three-dimensional simplex.

From Lemma 3, if $\hat{Q}_{\mathbf{k},T_s}$, $|\mathbf{k}| = n - 2$, is conditionally positive definite, then $p_n(\mathbf{x})$ is convex. Dahmen and Micchelli [8] pointed out that, for $s \leq 3$, if an $(s+1) \times (s+1)$ symmetric $A = [a_{ij}]_{1 \leq i,j \leq s+1}$ satisfies

$$a_{kk} + a_{ij} \geq a_{ik} + a_{kj}, \tag{30}$$

for all i, j, k, then A is conditionally positive definite. For $s > 3$, there exist matrices A satisfying (30) which are not conditionally positive definite. Obviously, for $A = \hat{Q}_{\mathbf{k},T_s}$, condition (30) is equivalent to

$$\Delta_{ki}\Delta_{kj}a_\alpha \geq 0. \tag{31}$$

Thus, by Lemma 3, Chang and Davis' condition (17) can be generalized to the three-dimensional setting, but it cannot be generalized to s-dimensional setting for $s > 3$. Condition (31) is also a necessary and sufficient convexity condition for the Bézier net of p_n(cf., [8]).

The following theorem gives a generalization of condition (18).

THEOREM 9. *The Bernstein-Bézier polynomial $p_n = B_n[\{a_\mathbf{k}\}; \lambda]$ is convex on T if its Bézier coefficients satisfy the condition*

$$\begin{aligned}
&\Delta_{10}\Delta_{10}a_\mathbf{k},\ \Delta_{20}\Delta_{20}a_\mathbf{k},\ \Delta_{30}\Delta_{30}a_\mathbf{k} \geq 0,\\
&(\Delta_{10}\Delta_{20}a_\mathbf{k})(\Delta_{12}\Delta_{12}a_\mathbf{k}) + (\Delta_{01}\Delta_{21}a_\mathbf{k})(\Delta_{02}\Delta_{12}a_\mathbf{k}) \geq 0,\\
&(\Delta_{20}\Delta_{30}a_\mathbf{k})(\Delta_{23}\Delta_{23}a_\mathbf{k}) + (\Delta_{02}\Delta_{32}a_\mathbf{k})(\Delta_{03}\Delta_{23}a_\mathbf{k}) \geq 0,\\
&(\Delta_{30}\Delta_{10}a_\mathbf{k})(\Delta_{31}\Delta_{31}a_\mathbf{k}) + (\Delta_{03}\Delta_{13}a_\mathbf{k})(\Delta_{01}\Delta_{31}a_\mathbf{k}) \geq 0,\\
&(\Delta_{21}\Delta_{01}a_\mathbf{k})(\Delta_{02}\Delta_{32}a_\mathbf{k})(\Delta_{03}\Delta_{13}a_\mathbf{k}) + (\Delta_{01}\Delta_{31}a_\mathbf{k})(\Delta_{10}\Delta_{30}a_\mathbf{k})(\Delta_{20}\Delta_{20}a_\mathbf{k})\\
&\quad + (\Delta_{03}\Delta_{23}a_\mathbf{k})(\Delta_{20}\Delta_{30}a_\mathbf{k})(\Delta_{10}\Delta_{10}a_\mathbf{k}) + (\Delta_{02}\Delta_{12}a_\mathbf{k})(\Delta_{10}\Delta_{20}a_\mathbf{k})(\Delta_{30}\Delta_{30}a_\mathbf{k})\\
&\quad - (\Delta_{01}\Delta_{21}a_\mathbf{k})(\Delta_{30}\Delta_{10}a_\mathbf{k})(\Delta_{20}\Delta_{30}a_\mathbf{k}) - (\Delta_{03}\Delta_{13}a_\mathbf{k})(\Delta_{20}\Delta_{30}a_\mathbf{k})(\Delta_{10}\Delta_{20}a_\mathbf{k})\\
&\quad - (\Delta_{02}\Delta_{32}a_\mathbf{k})(\Delta_{10}\Delta_{20}a_\mathbf{k})(\Delta_{30}\Delta_{10}a_\mathbf{k}) \geq 0.
\end{aligned} \tag{32}$$

In addition, (32) is also a necessary and sufficient convexity condition for $B_2[\{a_\mathbf{k}\}; \lambda]$ and $B_3[\{a_\mathbf{k}\}; \lambda]$. It is easy to see that condition (32) is equivalent to the positive semidefiniteness of $Q_{\mathbf{k},T_3}$, $|\mathbf{k}| = n-2$. Thus, from Lemma 2, we immediately obtain Theorem 9.

Now, we generalize the convexity condition in Theorem 4 to higher dimension setting. First, we point out that Theorem 4 can be proved in another way. Without loss of generality, we assume $u = 1$. Then, the $Q_{\mathbf{k},T}$ which satisfies (17) or (20) is positive semidefinite. Otherwise, $Q_{\mathbf{k},T}$ has at least one negative eigenvalue $\lambda < 0$. From $\det[Q_{\mathbf{k},T} - \lambda I] = 0$, we have

$$\Delta_\mathbf{k}^{(1)} + \Delta_\mathbf{k}^{(2)} < \Delta_\mathbf{k}^{(1)} + \Delta_\mathbf{k}^{(2)} - \lambda < \Delta_\mathbf{k}^{(1)} < |\Delta_\mathbf{k}^{(1)}|,$$

or

$$\Delta_\mathbf{k}^{(1)} + \Delta_\mathbf{k}^{(3)} < \Delta_\mathbf{k}^{(1)} + \Delta_\mathbf{k}^{(3)} - \lambda < \Delta_\mathbf{k}^{(1)} < |\Delta_\mathbf{k}^{(1)}|.$$

This contradiction shows that $Q_{\mathbf{k},T}$ must be positive semidefinite. Thus, $p_n(\mathbf{x})$ is convex. In the same manner, we can easily prove the following result.

THEOREM 10. *The Bernstein-Bézier polynomial $p_n = B_n[\{a_\mathbf{k}\}; \lambda]$ is convex on T_s if for any fixed $u \in \{0, 1, \dots, s\}$ its Bézier coefficients satisfies the condition*

$$(\Delta_{vu}\Delta_{vu}a_\mathbf{k}) \geq \sum_{\substack{w=0,1,\dots,s\\ w\neq u,v}} |\Delta_{vu}\Delta_{wu}a_\mathbf{k}|,$$

where $|\mathbf{k}| = n-2$, and $v = 0, 1, \dots, s$, $v \neq u$.

Sauer [9] gave some criteria of two different convexities for multivariate Bernstein-Bézier polynomials, one is stronger than the classical convexity, the other one is weaker.

3. MONOTONICITY AND POSITIVITY CRITERIA FOR BERNSTEIN-BÉZIER POLYNOMIALS

It is well known that a continuous piecewise polynomial function is monotone or positive if and only if all its polynomial pieces are monotone or positive, respectively. Thus, in this section, we consider the monotonicity and positivity criteria for Bernstein-Bézier polynomials over a simplex. Here, we are mostly interested in the criteria for quadratic Bernstein-Bézier polynomials.

We first discuss monotonicity criteria. Denote $\hat{a_{\mathbf{k}}} = (\triangle_{10}a_{\mathbf{k}}, \triangle_{20}a_{\mathbf{k}})$. From (14), we have the following lemma.

LEMMA 11. *Let $p_n \in \pi_n^2$ be a Bernstein-Bézier polynomial defined on $T = \langle \mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2 \rangle$, $\mathbf{x}^i = (x_i, y_i)$. Then*

$$\frac{\partial p_n(\mathbf{x}^i)}{\partial x} = n\mathbf{c}_{(1,0)}^{\mathsf{T}}\hat{a_{\mathbf{k}i}}, \qquad \frac{\partial p_n(\mathbf{x}^i)}{\partial y} = n\mathbf{c}_{(0,1)}^{\mathsf{T}}\hat{a_{\mathbf{k}i}}, \tag{33}$$

where $i = 0, 1, 2$, $\mathbf{k}_0 = (n-1, 0, 0)$, $\mathbf{k}_1 = (0, n-1, 0)$, $\mathbf{k}_2 = (0, 0, n-1)$, $\hat{a_{\mathbf{k}i}} = (\triangle_{10}a_{\mathbf{k}i}, \triangle_{20}a_{\mathbf{k}i})$, and

$$\mathbf{c}_{(1,0)} = \left(\frac{y_2 - y_0}{\delta_T}, \frac{y_0 - y_1}{\delta_T}\right)^{\mathsf{T}},$$
$$\mathbf{c}_{(0,1)} = \left(\frac{x_0 - x_2}{\delta_T}, \frac{x_1 - x_0}{\delta_T}\right)^{\mathsf{T}},$$

where δ_T is twice the area of the triangle T. Since

$$p(x, y) = \sum_{|\mathbf{i}|=2} a_{\mathbf{i}}\phi_{\mathbf{i}}^2(u, v, w),$$

where $a_{\mathbf{i}} = a_{i_0 i_1 i_2}$, $\mathbf{i} \in \mathbf{Z}_+^3$, $|\mathbf{i}| = 2$, we obtain

$$\begin{aligned} \frac{\partial p(x_i y_i)}{\partial x} &= \frac{2}{\delta_T}(\mathbf{a}_i \cdot \mathbf{b}), \\ \frac{\partial p(x_i y_i)}{\partial y} &= \frac{2}{\delta_T}(\mathbf{a}_i \cdot \mathbf{c}), \end{aligned} \tag{34}$$

$i = 0, 1, 2$, where δ_T is twice the area of the triangle T, and

$$\begin{aligned} \mathbf{a}_0 &= (a_{200}, a_{110}, a_{101}), \\ \mathbf{a}_1 &= (a_{110}, a_{020}, a_{011}), \\ \mathbf{a}_2 &= (a_{101}, a_{011}, a_{002}), \\ \mathbf{c} &= (x_3 - x_2, x_1 - x_3, x_2 - x_1), \\ \mathbf{b} &= (y_2 - y_3, y_3 - y_1, y_1 - y_2). \end{aligned}$$

Since $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ are linear functions defined on T, p is monotone nondecreasing (or nonincreasing) in both of the directions along the x-axis and the y-axis if and only if $\frac{\partial p(x_i y_i)}{\partial x}$ and $\frac{\partial p(x_i y_i)}{\partial y}$, $i = 0, 1, 2$, are nonnegative (or nonpositive). Thus, we have the following theorem.

THEOREM 12. *A quadratic Bernstein-Bézier polynomial p defined on a triangular domain T with vertices (x_i, y_i), $i = 0, 1, 2$, is nondecreasing (or nonincreasing) in both x-direction and y-direction if and only if its Bézier coefficients $a_{i_0 i_1 i_2}$, $\mathbf{i} \in \mathbf{Z}_+^3$, $|\mathbf{i}| = 2$, satisfy the linear inequalities*

$$K\mathbf{a} \geq 0, \qquad (\text{or } K\mathbf{a} \leq 0), \tag{35}$$

where $\mathbf{a} = (a_{200}, a_{110}, a_{020}, a_{011}, a_{002}, a_{101})$, and

$$K = \begin{bmatrix} y_1 - y_2 & y_2 - y_0 & 0 & 0 & 0 & y_0 - y_1 \\ x_2 - x_1 & x_0 - x_2 & 0 & 0 & 0 & x_1 - x_0 \\ 0 & y_1 - y_2 & y_2 - y_0 & y_0 - y_1 & 0 & 0 \\ 0 & x_2 - x_1 & x_0 - x_2 & x_1 - x_0 & 0 & 0 \\ 0 & 0 & 0 & y_2 - y_0 & y_0 - y_1 & y_1 - y_2 \\ 0 & 0 & 0 & x_0 - x_2 & x_1 - x_0 & x_2 - x_1 \end{bmatrix}. \tag{36}$$

Next, we will give the positivity criteria of a quadratic Bernstein-Bézier polynomial p defined on T. By using the notations in (27) and (28), we have

$$\frac{\partial p}{\partial u} = 2A_{101}u + 2B_{002}v + 2(a_{101} - a_{002}),$$
$$\frac{\partial p}{\partial v} = 2B_{002}u + 2A_{011}v + 2(a_{011} - a_{002}).$$

If $\det Q_T = A_{101}A_{011} - B_{002}^2 \neq 0$, we obtain a critical point $(u_0, v_0, 1-u_0-v_0)$ of $p(u, v, 1-u-v)$, where

$$u_0 = \frac{[A_{011}(a_{002} - a_{101}) - B_{002}(a_{002} - a_{011})]}{\det Q_T},$$
$$v_0 = \frac{[A_{101}(a_{002} - a_{011}) - B_{002}(a_{002} - a_{101})]}{\det Q_T}. \tag{37}$$

Also, for $\overline{p}(u, v, w) = p(x, y)$, we have

$$\overline{p}(u_0, v_0, 1 - u_0 - v_0) = \frac{1}{\det Q_T}\left[a_{200}a_{020}a_{002} + 2a_{110}a_{011}a_{101} - a_{110}^2 a_{002} - a_{011}^2 a_{200} - a_{101}^2 a_{020}\right]. \tag{38}$$

It is obvious that $(u_0, v_0, 1 - u_0 - v_0) \in T$ if and only if

$$u_0 \geq 0, \qquad v_0 \geq 0 \qquad \text{and} \qquad \det Q_T \geq u_0 + v_0. \tag{39}$$

Denote

$$\begin{aligned} F_1(a) &= A_{011}(a_{002} - a_{101}) - B_{002}(a_{002} - a_{011}), \\ F_2(a) &= A_{101}(a_{002} - a_{011}) - B_{002}(a_{002} - a_{101}), \\ F_3(a) &= \det Q_T - [A_{011}(a_{002} - a_{101}) - B_{002}(a_{002} - a_{011})] \\ &\quad - [A_{101}(a_{002} - a_{011}) - B_{002}(a_{002} - a_{101})]. \end{aligned} \tag{40}$$

Thus, (39) is equivalent to

$$F_j(a) \geq 0, \qquad j = 1, 2, 3.$$

Since on the vertices and edges of T we have

$$p(1,0,0) = a_{200}, \qquad p(0,1,0) = a_{020}, \qquad p(0,0,1) = a_{002},$$
$$p(0, v, 1-v) = (a_{020} + a_{002} - 2a_{011})v^2 + 2(a_{011} - a_{002})v + a_{002},$$
$$p(u, 0, 1-u) = (a_{200} + a_{002} - 2a_{101})u^2 + 2(a_{101} - a_{002})u + a_{002},$$
$$p(u, 1-u, 0) = (a_{200} + a_{020} - 2a_{110})u^2 + 2(a_{110} - a_{020})u + a_{020},$$

we conclude that p is nonnegative on the boundary of T if and only if

$$a_{200},\ a_{020},\ a_{002} \geq 0,$$

and

$$a_{011} \geq -\sqrt{a_{020}a_{002}}, \qquad a_{101} \geq -\sqrt{a_{200}a_{002}}, \qquad a_{110} \geq -\sqrt{a_{200}a_{020}}.$$

If $F_j(a) > 0$, $j = 1, 2, 3$, then

$$\det Q_T \geq F_1(a) + F_2(a) > 0,$$

and a critical point $(u_0, v_0, 1 - u_0 - v_0)$ exists in T, and $p(u_0, v_0, 1 - u_0 - v_0)$ is shown as in (37),(38). If one of the $F_j(a)$, j=1,2,3, is nonpositive, then $p(u, v, w)$ must attain its minimum on the boundary of T. Thus, we have the following.

PROPOSITION 13. *Let p be a quadratic Bernstein-Bézier polynomial defined on T, and let $F_j(a)$, $j = 1, 2, 3$ be the expressions defined as in* (40). *If $F_j(a) > 0$, $j = 1, 2, 3$, then p is nonnegative if and only if*

(i) $a_{200}, a_{020}, a_{002} \geq 0$,

(ii) $a_{011} \geq -\sqrt{a_{020}a_{002}}$, $\quad a_{101} \geq -\sqrt{a_{200}a_{002}}$, $\quad a_{110} \geq -\sqrt{a_{200}a_{020}}$, *and*

(iii) $a_{200}a_{020}a_{002} + 2a_{110}a_{011}a_{101} - a_{110}^2 a_{002} - a_{011}^2 a_{200} - a_{101}^2 a_{020} \geq 0$

hold. If one of the $F_j(a) \leq 0$, j=1,2,3, then p is nonnegative if and only if both (i) *and* (ii) *hold. Obviously, if in Proposition 13 we change nonnegativity to positivity, then it still holds when every inequality sign is changed to strict inequality.*

Proposition 13 was obtained by Hadeler by using the properties of copositive matrices (cf., [10–12]).

Assume $a_{200}, a_{020}, a_{002} > 0$, and denote

$$\begin{aligned} r_1 &= \frac{a_{110}}{\sqrt{a_{200}a_{020}}}, & r_2 &= \frac{a_{011}}{\sqrt{a_{020}a_{002}}}, & r_3 &= \frac{a_{101}}{\sqrt{a_{200}a_{002}}}, \\ \hat{u} &= \frac{u}{\sqrt{a_{020}a_{002}}}, & \hat{v} &= \frac{v}{\sqrt{a_{200}a_{002}}}, & \hat{w} &= \frac{w}{\sqrt{a_{200}a_{020}}}. \end{aligned} \tag{41}$$

Then, $\overline{p}(u, v, w) = p(x, y)$ can be written as

$$\overline{p}(u, v, w) = a_{200}a_{020}a_{002}\tilde{p}(\hat{u}, \hat{v}, \hat{w}),$$

where

$$\tilde{p}(\hat{u}, \hat{v}, \hat{w}) = \hat{u}^2 + \hat{v}^2 + \hat{w}^2 + 2r_1\hat{u}\hat{v} + 2r_2\hat{v}\hat{w} + 2r_3\hat{u}\hat{w}.$$

If $\tilde{p}$ is considered as a linear function $\hat{p}$ of r_1, r_2, and r_3, then it is easy to see $\hat{p}$ is positive if $r_1, r_2, r_3 \geq -1$ (cf., (ii) in Proposition 13) and $r_1 + r_2 + r_3 \geq -1$. In fact, consider the solution of the linear programming problem

$$\begin{aligned} \hat{p}^* = \min_{r_1, r_2, r_3} \hat{p}(r_1, r_2, r_3) \\ \text{subject to } r_1, r_2, r_3 \geq -1 \\ r_1 + r_2 + r_3 \geq -1. \end{aligned}$$

It is solved by

$$\hat{p}^* = \min\{\hat{p}(-1, 0, 0), \hat{p}(0, -1, 0), \hat{p}(0, 0, -1)\},$$

and

$$\begin{aligned} \hat{p}(-1, 0, 0) &= \hat{u}^2 + \hat{v}^2 + \hat{w}^2 - 2\hat{u}\hat{v} = (\hat{u} - \hat{v})^2 + \hat{w}^2 \geq 0, \\ \hat{p}(0, -1, 0) &= \hat{u}^2 + (\hat{v} - \hat{w})^2 \geq 0, \\ \hat{p}(0, 0, -1) &= (\hat{u} - \hat{w})^2 + \hat{v}^2 \geq 0. \end{aligned}$$

Thus, if $a_{200}, a_{020}, a_{002} > 0$, $r_1, r_2, r_3 \geq -1$ and $r_1 + r_2 + r_3 \geq -1$, then $\overline{p}(u, v, w)$ is nonnegative.

On the other hand, if $\overline{p}(u, v, w)$ is nonnegative on T, and $a_{200}, a_{020}, a_{002} > 0$, then we may take

$$(u, v, w) = \left(\frac{\sqrt{a_{020}a_{002}}}{a}, \frac{\sqrt{a_{200}a_{002}}}{a}, \frac{\sqrt{a_{200}a_{020}}}{a} \right) \in T,$$

where

$$a = \sqrt{a_{200}a_{020}} + \sqrt{a_{020}a_{002}} + \sqrt{a_{200}a_{002}},$$

so that

$$p = \frac{a_{200}a_{020}a_{002}}{a}(3 + 2r_1 + 2r_2 + 2r_3) \geq 0.$$

Thus,

$$r_1 + r_2 + r_3 \geq -\frac{3}{2}.$$

We obtain the following results.

PROPOSITION 14. *Let p be a quadratic Bernstein-Bézier polynomial defined on T with $a_{200}, a_{020}, a_{002} > 0$, and let the corresponding r_1, r_2, and r_3 of p be defined as in* (41). *If $r_1, r_2, r_3 \geq -1$ and $r_1 + r_2 + r_3 \geq -1$, then p is nonnegative. If p is nonnegative, then $r_1, r_2, r_3 \geq -1$ and $r_1 + r_2 + r_3 \geq -3/2$.*

It is easy to see, that if $r_1, r_2, r_3 \geq -1$ and $r_1 + r_2 + r_3 < -1$, then

$$a_{011}a_{110} - a_{020}a_{101} > 0,$$
$$a_{011}a_{101} - a_{002}a_{110} > 0,$$

and

$$a_{020}a_{002} - {a_{011}}^2 \geq 0;$$

that is,

$$F_1(a) > 0.$$

Similarly, we have $F_2(a) > 0$ and $F_3(a) > 0$. Thus, from Propositions 13 and 14, since the left hand side of (iii) in Proposition 13 can be written as $1 + 2r_1r_2r_3 - r_1^2 - r_2^2 - r_3^2$, we have the following theorem.

THEOREM 15. *Let p be a quadratic Bernstein-Bézier polynomial defined on T with $a_{200}, a_{020}, a_{002} > 0$, and the corresponding r_1, r_2, r_3 of p be defined as in* (41). *If $r_1 + r_2 + r_3 < -1$, then p is nonnegative if and only if*

(i) $r_1, r_2, r_3 \geq -1$, and
(ii) $1 + 2r_1r_2r_3 - r_1^2 - r_2^2 - r_3^2 \geq 0$

hold. If $r_1 + r_2 + r_3 \geq -1$, then p is nonnegative if and only if condition (i) *holds.*

Unfortunately, all of the positivity criteria shown above are nonlinear. For the sake of convenience in application, we try to find a linear positivity criterion. One idea is to apply Lemma 1. In fact, we have the following results.

THEOREM 16. *Let $B_n[\{a_{\mathbf{k}}\};\lambda]$ be a Bernstein-Bézier polynomial defined on T. If its Bézier coefficients satisfy either*

$$a_{n\mathbf{e}^i} + (n-1)! \sum_{\substack{|\mathbf{k}|=n \\ \mathbf{k}\neq n\mathbf{e}^0,\dots,n\mathbf{e}^s \\ a_{\mathbf{k}}<0}} \frac{k_i}{\mathbf{k}!} a_{\mathbf{k}} \geq 0, \tag{42}$$

for $i = 0,1,\dots,s$, or

$$a_{n\mathbf{e}^i} + (n-1)! \sum_{\substack{|\mathbf{k}|=n \\ \mathbf{k}\neq n\mathbf{e}^0,\dots,n\mathbf{e}^s \\ a_{\mathbf{k}}<0}} \left(\frac{2}{n}\right)^{n-1} \frac{k_i^n}{\mathbf{k}!} a_{\mathbf{k}} \geq 0, \tag{43}$$

for $i = 0,1,\dots,s$, where $\mathbf{e}^i$ is the i^{th} coordinate vector, then $B_n[\{a_{\mathbf{k}}\};\lambda] \geq 0$.

REMARK. (42) was also given in the three dimension setting by Wang and Liu [13].

PROOF. We first prove condition (42). From (4), we have

$$a_{\mathbf{k}}\phi_{\mathbf{k}}^n(\lambda) \geq a_{\mathbf{k}} \frac{(n-1)!}{k!} \sum_{i=0}^{s} k_i \lambda_i^n,$$

for any $a_{\mathbf{k}} < 0$. Thus,

$$\begin{aligned} B_n[\{a_{\mathbf{k}}\};\lambda] &= \sum_{|\mathbf{k}|=n} a_{\mathbf{k}}\phi_{\mathbf{k}}^n(\lambda) \\ &\geq \sum_{i=0}^{s} a_{n\mathbf{e}^i}\lambda_i^n + \sum_{\substack{|\mathbf{k}|=n \\ \mathbf{k}\neq n\mathbf{e}^0,\dots,n\mathbf{e}^s \\ a_{\mathbf{k}}<0}} a_{\mathbf{k}}\phi_{\mathbf{k}}^n(\lambda) \\ &\geq \sum_{i=0}^{s} \left(a_{n\mathbf{e}^i} + \sum_{\substack{|\mathbf{k}|=n \\ \mathbf{k}\neq n\mathbf{e}^0,\dots,n\mathbf{e}^s \\ a_{\mathbf{k}}<0}} \frac{(n-1)!k_i}{\mathbf{k}!} a_{\mathbf{k}} \right) \lambda_i^n. \end{aligned}$$

Hence, if (42) holds, then $B_n[\{a_{\mathbf{k}}\};\lambda] \geq 0$.

Next, we prove condition (43). From (3), we have

$$a_{\mathbf{k}}\phi_{\mathbf{k}}^n(\lambda) \geq a_{\mathbf{k}} \frac{(n-1)!}{k!n^{n-1}} \left(\sum_{j=0}^{s} k_j\lambda_j \right)^n,$$

for any $a_{\mathbf{k}} < 0$. From (4), it follows that

$$\begin{aligned}
B_n[\{a_{\mathbf{k}}\};\lambda] &= \sum_{|\mathbf{k}|=n} a_{\mathbf{k}} \phi_{\mathbf{k}}^n(\lambda) \\
&\geq \sum_{i=0}^{s} a_{n\mathbf{e}^i} \lambda_i^n + \sum_{\substack{|\mathbf{k}|=n \\ \mathbf{k}\neq n\mathbf{e}^0,\dots,n\mathbf{e}^s \\ a_{\mathbf{k}}<0}} a_{\mathbf{k}} \phi_{\mathbf{k}}^n(\lambda) \\
&\geq \sum_{i=0}^{s} a_{n\mathbf{e}^i} \lambda_i^n + \sum_{\substack{|\mathbf{k}|=n \\ \mathbf{k}\neq n\mathbf{e}^0,\dots,n\mathbf{e}^s \\ a_{\mathbf{k}}<0}} a_{\mathbf{k}} \frac{(n-1)!}{\mathbf{k}! n^{n-1}} \sum_{|\beta|=n} \phi_{\beta}^n(\lambda \mathbf{k}) \\
&\geq \sum_{i=0}^{s} a_{n\mathbf{e}^i} \lambda_i^n + \sum_{\substack{|\mathbf{k}|=n \\ \mathbf{k}\neq n\mathbf{e}^0,\dots,n\mathbf{e}^s \\ a_{\mathbf{k}}<0}} a_{\mathbf{k}} \frac{(n-1)!}{\mathbf{k}! n^{n-1}} \sum_{|\beta|=n} \frac{(n-1)!}{\beta!} \sum_{i=0}^{s} \beta_i (\lambda_i k_i)^n \\
&\geq \sum_{i=0}^{s} a_{n\mathbf{e}^i} \lambda_i^n + \sum_{\substack{|\mathbf{k}|=n \\ \mathbf{k}\neq n\mathbf{e}^0,\dots,n\mathbf{e}^s \\ a_{\mathbf{k}}<0}} a_{\mathbf{k}} \frac{(n-1)! k_i^n}{\mathbf{k}! n^{n-1}} \lambda_i^n \left(\sum_{|\mathbf{k}|=n} \frac{(n-1)!}{\beta!} \beta_i \right) \\
&\geq \sum_{i=0}^{s} \left(a_{n\mathbf{e}^i} + \sum_{\substack{|\mathbf{k}|=n \\ \mathbf{k}\neq n\mathbf{e}^0,\dots,n\mathbf{e}^s \\ a_{\mathbf{k}}<0}} \frac{(n-1)! k_i^n}{\mathbf{k}!} \left(\frac{2}{n}\right)^{n-1} a_{\mathbf{k}} \right) \lambda_i^n.
\end{aligned}$$

Hence, (43) holds and $B_n[\{a_{\mathbf{k}}\};\lambda] \geq 0$. If $k_i \leq n/2$; i.e.,

$$\frac{2k_i}{n} \geq \left(\frac{2k_i}{n}\right)^n,$$

then condition (43) is weaker than condition (42); otherwise condition (42) is weaker than (43).

Another way to obtain a linear positivity criterion is to use mid-point subdivisions of the original triangle as shown in Section 2. Obviously, if the Bézier coefficients of the restrictions of p_n on each subtriangle are positive, then p_n is positive on the original triangle. On the other hand, as we have pointed out, by successively subdividing the original triangle T often enough, the Bézier coefficients of the restricted p_n on the subtriangles are very close to the values of p_n at the nodes of the discrete subtriangles. Thus, if the subtriangles are small enough, the above idea will give a sufficient positivity condition in good accuracy; meanwhile, this condition is "almost" necessary [14].

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